$e^+e^- \to \gamma\,\gamma^*$ at two loops

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Workshop on Radiative Corrections and Monte Carlo simulations for electron-positron collisions
Scuola Normale Superiore
May 8, 2025

LEVERHULME TRUST

This talk: I plan to present a status of the higher-order corrections required in `Phokhara` at NNLO.

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From Graziano's talk

Desiderable:

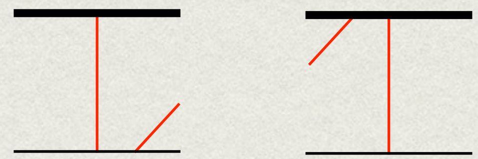
CODE	mmg	ppg	Comments (matrix element, FSC)
Phokhara	NNLO	NNLO	exponentiation, FxsQED, GVMD,FsQED

- Fixed order NLO + soft photon resummation (see Jeremy's talk)
- GVMD (NLO) and $F \times$ sQED (NNLO) within Phokhara (see Pau's talk)
- Fixed order NNLO :: in the making —> first look at Initial State Content (This talk)

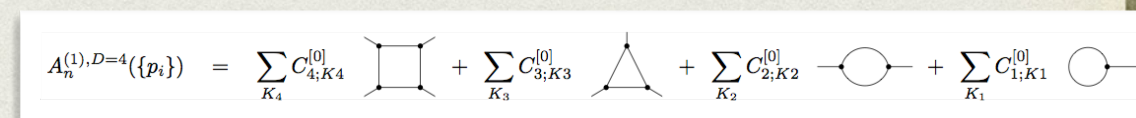
$e^+e^- \to F^+F^-\gamma$ @ NNLO

Anatomy @ LO

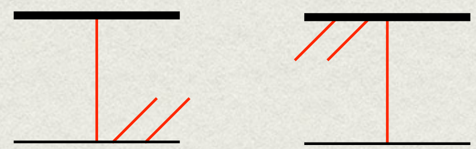
- Born matrix element
  tree-level & n-pt process



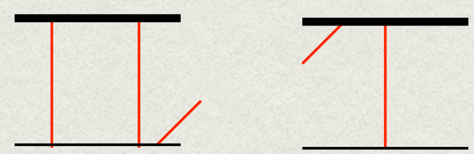
Anatomy @ NLO



- Real contribution tree-level $(n+1)$-particles



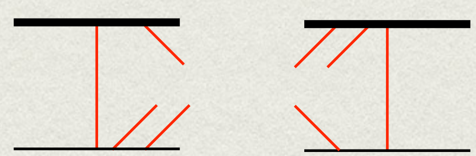
- Virtual Contribution one-loop $(n+1)$-particles



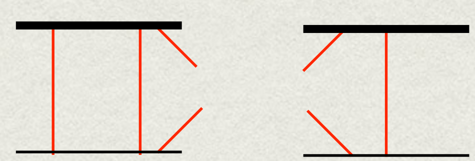
- ☑ Automated one-loop Feynman integral & phase-space evaluation
- ☑ IR subtraction schemes under control
- ☑ Efficient numerical evaluation (MC friendly)

Anatomy @ NNLO

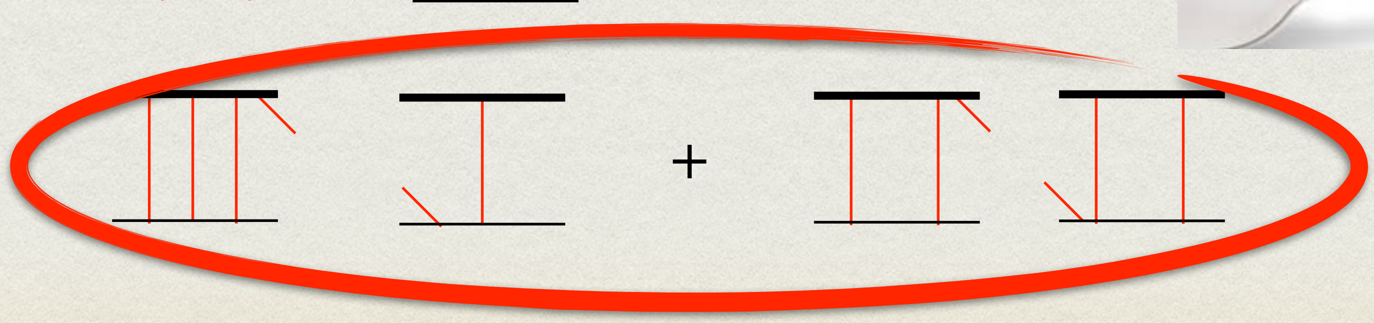
- Real-Real contribution
  Tree-level $(n+2)$-particles



- Real-Virtual Contribution
  one-loop $(n+1)$-particles

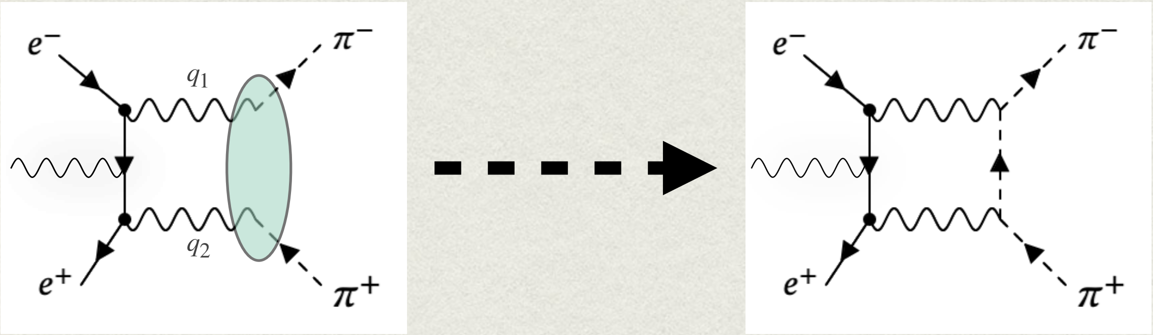


- Virtual-Virtual Contribution
  two-loop n-particles



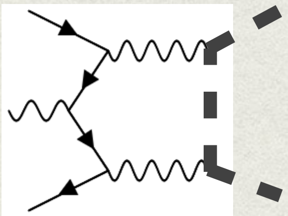
- ☐ Harder (but doable) phase-space integration
- ☐ Extend numerical evaluation of one-loop Feynman integrals
- ☐ Basis of two-loop Feynman integral not known

$e^+e^- \to \pi\pi\gamma$ @ 1L



$$\times \left[F(M_{\pi\pi}) + F(q_1^2)F(q_2^2) - F(M_{\pi\pi})\right]$$

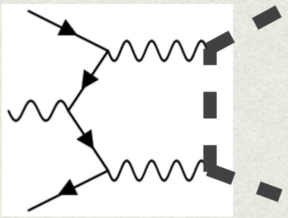
- $F \times$ sQED



$$+\text{boxes} + \text{crossed diagrams} = \frac{c_{-1}}{\epsilon} + c_0 + c_1\epsilon + \mathcal{O}(\epsilon^2)$$

(included in Phokhara)

- GVMD



$$+\text{boxes} + \text{crossed diagrams} = \tilde{c}_0 + \mathcal{O}(\epsilon)$$

(to be included in Phokhara)

$e^+e^- \rightarrow \pi\pi\gamma$ @ 2L

Amplitude generation

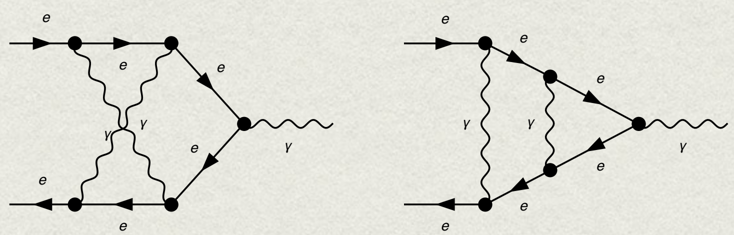
Algebraic decomposition

Loop integral evaluation

MC input

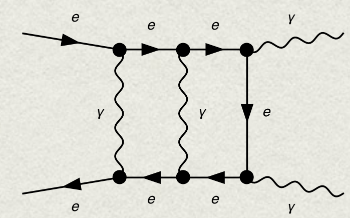
- Two-loop gauge invariant pieces

$f^+f^- \rightarrow \gamma^* \rightarrow F^+F^- + \gamma$



Easy (m_f^2, s)

$f^+f^- \rightarrow \gamma\,\gamma^* \rightarrow F^+F^-$

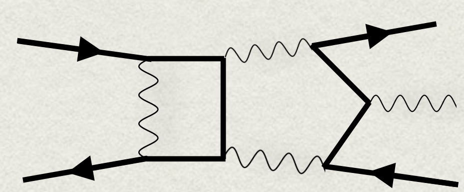


$+\ldots$

"Normal" (s, t, m_e^2, q^2)

Progress on these Feyman integrals

$f^+f^- \rightarrow F^+F^-\gamma$



$+\ldots$

Hard $(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, m_f^2, m_F^2)$

Efficient evaluation of Scattering Amplitudes

Radiative return processes @ NNLO

- Evaluation of Feynman integrals by the method of differential equations

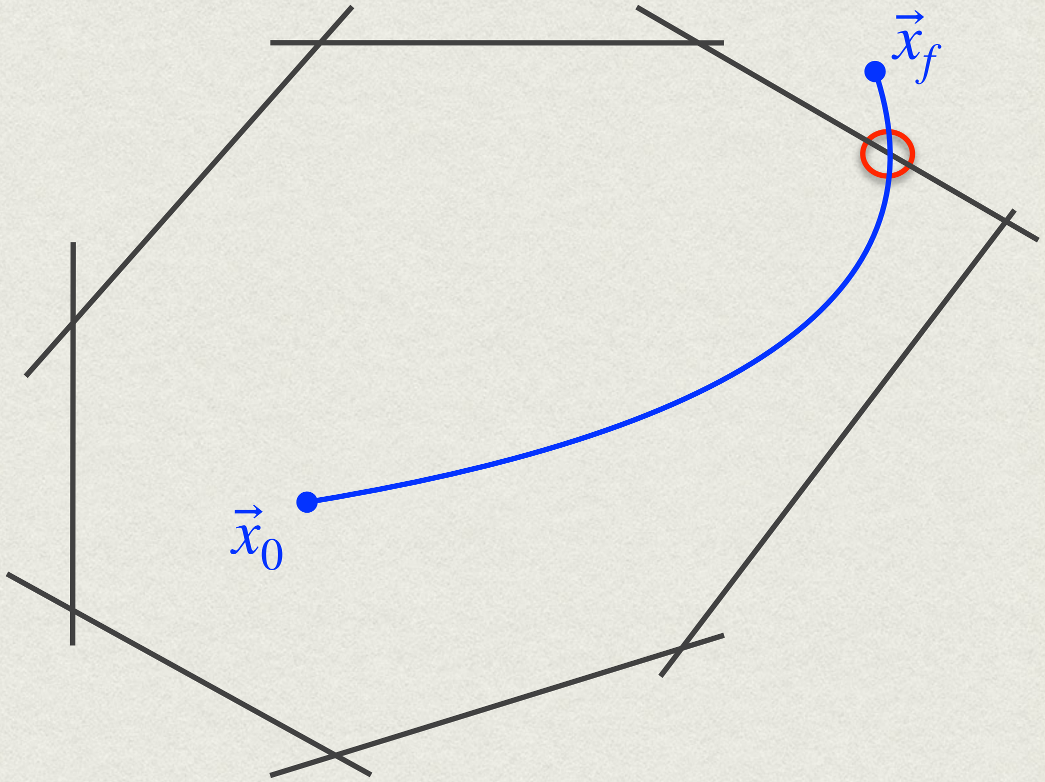
$$\partial_x \vec{I}(\vec{x};\epsilon) = A_x(\vec{x};\epsilon)\,\vec{I}(\vec{x};\epsilon)$$

$\downarrow$ $\vec{x}$ (kinematic invariants)

$$d\vec{J}(\vec{x};\epsilon) = \epsilon\, d\tilde{A}(\vec{x})\,\vec{J}(\vec{x};\epsilon)$$

$\downarrow$

$$\vec{J}(\vec{x};\epsilon) = \mathscr{P}\exp\left(\epsilon \int_\gamma d\tilde{A}\right)\vec{J}_0$$



- ☑ When possible find a canonical basis $\vec{J} = R\vec{I}$ **[Henn 2013]**
- ☑ Solve DEQ along the path **[Moriello 2019]**
- ☑ Get boundary constants $\vec{J}_0$ analytically or numerically
- ☑ Account for analytic continuations when crossing regions

- Currently working on C++ implementation
  Input :: DEQ+ boundary constants

Feynman integrals in terms of graded functions

$$d\vec{J} = \epsilon \, d\tilde{A} \, \vec{J}$$

$J_i^{(k)}$ are redundant order-by-order in ϵ

[Henn (2013)]

$$\downarrow$$

$$\overrightarrow{W} = \mathbb{R} \, \vec{J}$$

W_{i_k} are independent integrals order-by-order in ϵ

[Chicherin, Sotnikov, Zoia (2021)]

[Gehrmann, Henn, Jakubčík, Lim, Mella, Syrrakos, Tancredi, WJT (2024)]

$$\downarrow$$

$$W_{i_k}\left(\vec{x};\epsilon\right) = \sum_{k'=k}^{4} \epsilon^{k'} \, w_{i_k}^{(k')}\left(\vec{x}\right)$$

$w_{i_k}^{(k')}$ are independent transcendental functions

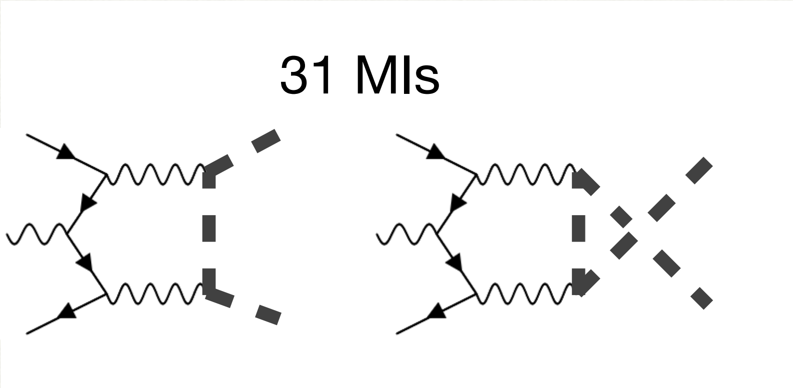
$$\downarrow$$

$$dw_{i_k}^{(k')}\left(\vec{x}\right) = \sum_{l=0}^{4} d\Omega_{i_k i_l} \, w_{i_l}^{(k'-1)}$$

Differential equations independent of ϵ

[Henn, Caron-Huot (2014)]

Gauge invariant combination of pentagon w/ boxes



Presence of 13 square roots

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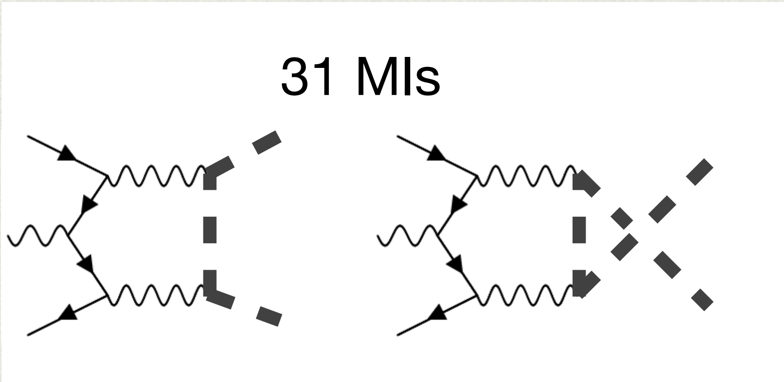

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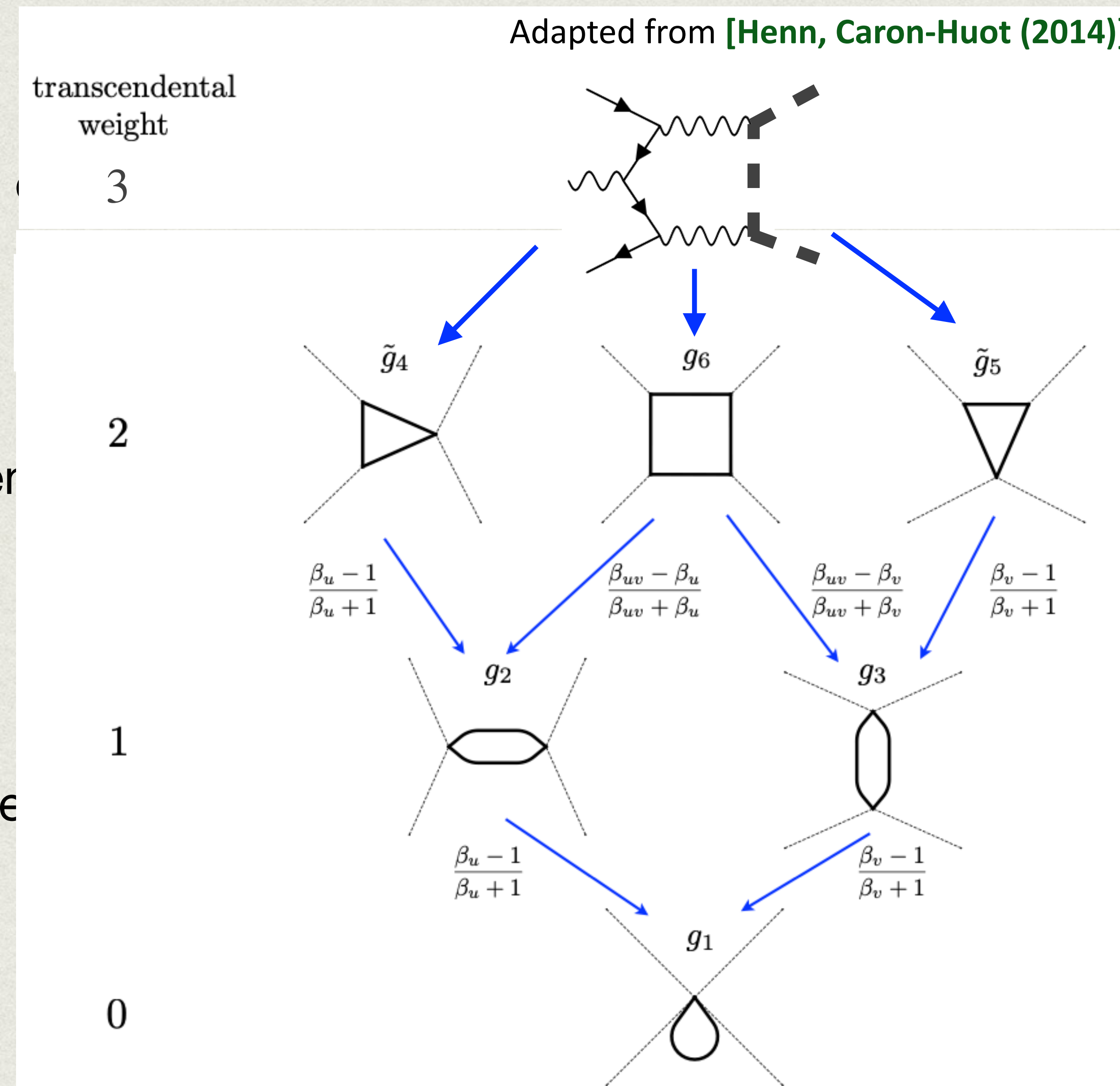

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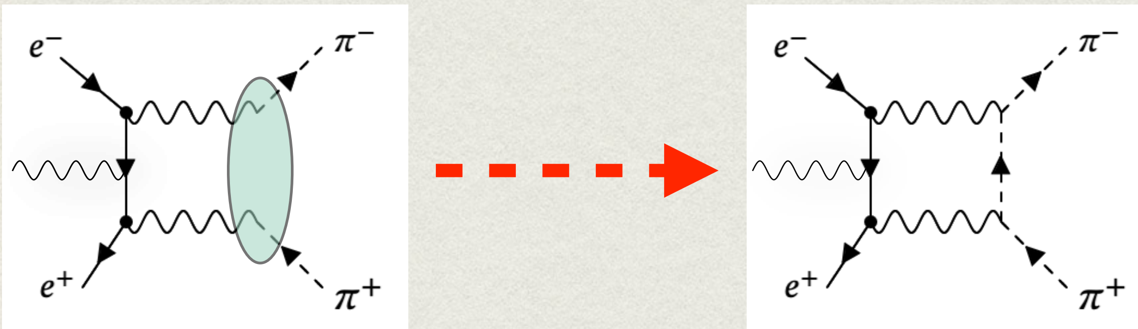
Radiative return processes @ NNLO



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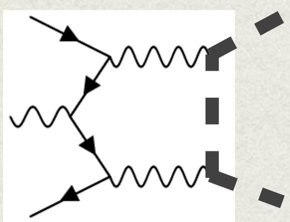
- $F \times$ sQED



$$+\text{boxes} + \text{crossed diagrams} = \frac{c_{-1}}{\epsilon} + c_0 + c_1\epsilon + \mathcal{O}(\epsilon^2)$$

$$c_{-1} = \left|A^{(0)}_{\text{ISC}}\right|^2 \left[-\frac{2w_1^{(1)}\left(m_e^2 + m_\pi^2 - s_{15}\right)}{r_8} - \frac{2w_2^{(1)}\left(m_e^2 + m_\pi^2 - s_{23}\right)}{r_9} + (3 \leftrightarrow 5)\right]$$

$$c_{0|1} = \sum_{ij} r_{ij}\, w_i^{(j)}$$

31 functions present in c_0

54 functions present in c_1

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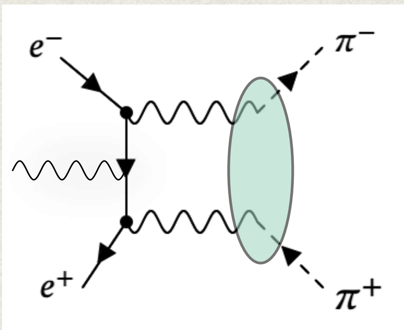
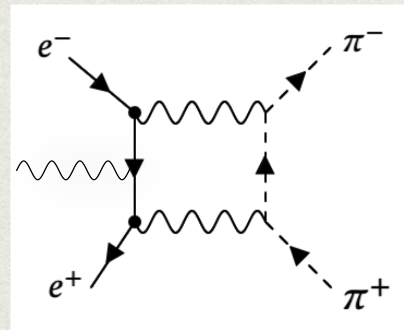
Radiative return processes @ NNLO





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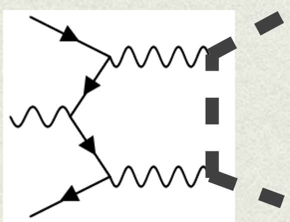

$$+\text{boxes} + \text{crossed diagrams} = \frac{c_{-1}}{\epsilon} + c_0 + c_1\epsilon + \mathcal{O}(\epsilon^2)$$

Log's :: exactly match IR pole prediction

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Progress on ISR

$e^+e^- \to \pi\pi\gamma$ @ 2L

- Insights from "massive" calculations of Feynman integrals

  - $e^+e^- \to e^+e^-$ with $m_e^2 \neq 0$ [Henn, Smirnov (2013), Duhr et al (2021, 2023)]
  - $e^+e^- \to \mu^+\mu^-$ with $m_e^2 \neq m_\mu^2 \neq 0$ [Heller (2021)]

- Practical approach

  - Use canonical DEQs as much as possible
  - Find integrals that obeys partial DEQs with the form

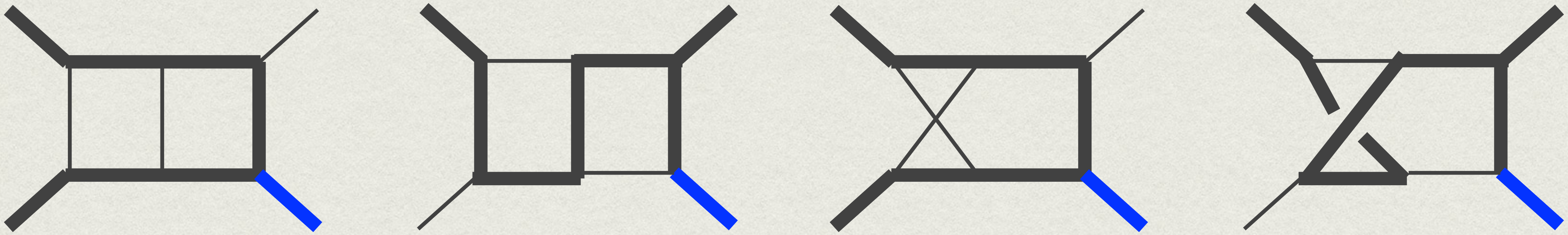
$$\frac{\partial \vec{J}}{\partial x} = \sum_{k=0}^{2} \epsilon^k A_k \vec{J}$$

(move 'difficult' integrals to very late stages)

  - Make use of our C++ implementation

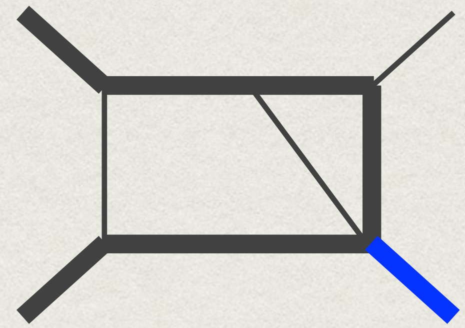
$e^+e^- \to \pi\pi\gamma$ @ 2L

- Main focus on ISC @ 2L $(f^+f^- \to \gamma\,\gamma^*)$



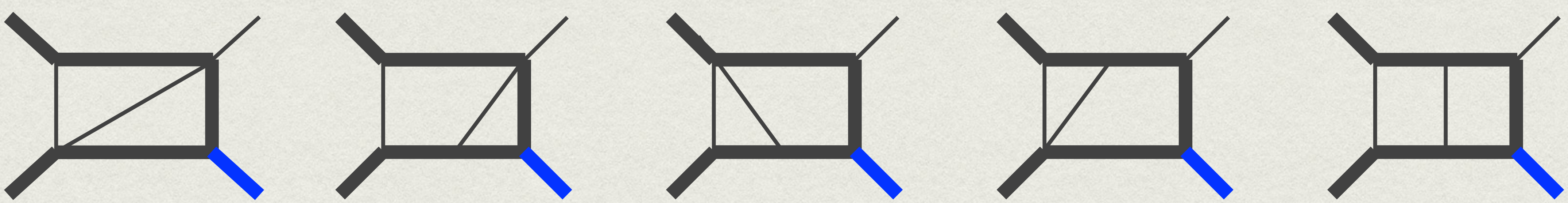
$+\ldots$

- 50 MIs obey canonical DEQ w/ 3 square roots



$d\vec{J} = \epsilon\, d\tilde{A}\, \vec{J}$

- Elliptic sectors

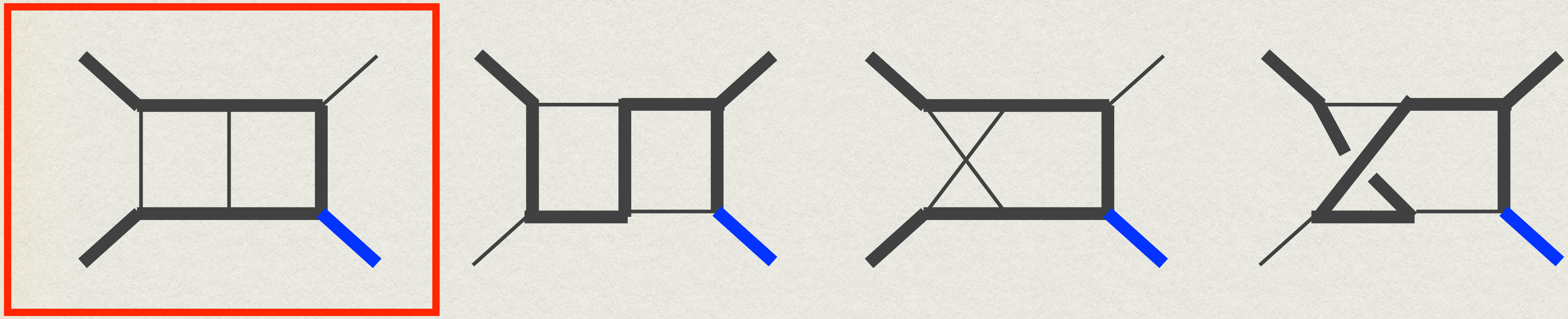


6 MIs | 3 MIs | 3 MIs | 3 MIs | 3 MIs

Currently looking for integrals sector-by-sector

$e^+e^- \to \pi\pi\gamma$ @ $2L$

- Main focus on ISC @ 2L $(f^+f^- \to \gamma\,\gamma^*)$



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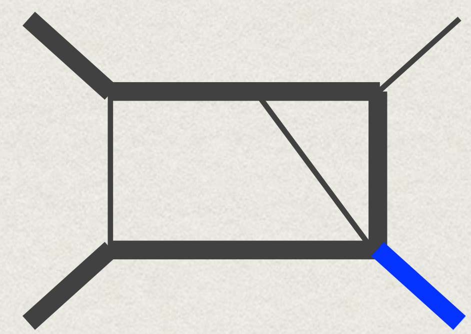

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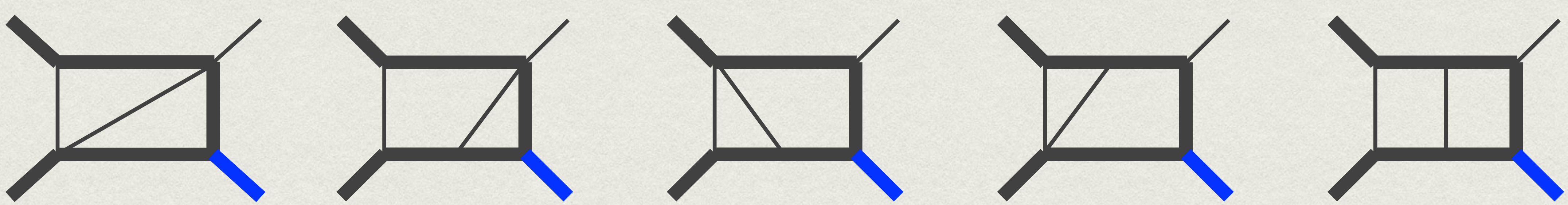

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Conclusions

- We have reached:

  - ☑ Feynman integrations in terms of graded functions

    * Analytic results for $e^+e^- \to \pi\pi\gamma$

  - ☑ First look at the evaluation of two-loop Feynman integrals for $e^+e^- \to \gamma\,\gamma^*$
  - ☑ Numerical evaluation of first 4-pt subsector

- Open questions & future directions

  - ☐ Implement GVMD within Phokhara
  - ☐ Extend to FSC of $e^+e^- \to \pi\pi\gamma$
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